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Christoffel type functions for *m*-orthogonal polynomials

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Abstract

The complete extension of the Christoffel type functions to the *m*-orthogonal polynomials is established. The properties of the Christoffel type functions are investigated. The estimations and asymptotics of the Christoffel type functions for some weights including a generalized Jacobi weight are also given.

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1. Introduction and definition

We denote by $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{N}_1$, or $\mathbf{N}_2$ the set of positive, nonnegative, odd integers, or even integers, respectively. We also denote by $\mathbf{R}$ the set of real numbers.

Let μ be a nondecreasing function on $\mathbf{R}$ with infinitely many points of increase such that all moments of $d\mu$ are finite. We call $d\mu$ a measure. If μ happens to be absolutely continuous then we will usually write w instead of μ' and will call w a weight. The support of $d\mu$ is the set of points of increase of $\mu(x)$ and the smallest interval containing it is denoted by $\Delta(d\mu)$. The symbol $\mathbf{P}_N$ stands for the set of algebraic polynomials of degree at most N. The symbol ∂P denotes the exact degree of the polynomial $P \neq 0$, i.e., $P \in \mathbf{P}_{\partial P} \setminus \mathbf{P}_{\partial P-1}$.

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We denote by $c, c_1, \ldots$ positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas. We write $a_n \sim b_n$ if $c_1 \leqslant a_n/b_n \leqslant c_2$ holds for every n. The notations $a(x) \sim b(x)$ and $a_n(x) \sim b_n(x)$ have similar meaning.

Throughout this paper let $m \in \mathbf{N}$ $(m \geqslant 2)$, $\mathbf{M}_1 = \{j \leqslant m-3 : m-j \in \mathbf{N}_1\}$, and $\mathbf{M}_2 = \{j \leqslant m-2 : m-j \in \mathbf{N}_2\}$. Put $\mathbf{P}_N^* = \{P(x) = c(x-y_1)\cdots(x-y_r) : c, y_1, \ldots, y_r \in \mathbf{R},\ r \leqslant N\}$ and $\mathbf{P}_N^*(x) = \{P \in \mathbf{P}_N^* : P(x) = 1\}$ for $x \in \mathbf{R}$. We agree $\mathbf{P}_0^* = \mathbf{P}_0$.

We define the m-monic orthogonal polynomials

$$P_n(d\mu, m; x) = x^n + \cdots, \quad n = 0, 1, \ldots,$$

for which

$$\int_{\mathbf{R}} |P_n(d\mu, m; x)|^m \, d\mu(x) = \min_{P(x)=x^n+\cdots} \int_{\mathbf{R}} |P(x)|^m \, d\mu(x). \tag{1.1}$$

If $\mu' = w$ is a weight then we will usually write $P_n(w, m; x)$ instead of $P_n(d\mu, m; x)$. According to Theorem 4 in [2] (see also Theorem 2.2 here), if $x_k = x_{kn}(d\mu, m)$ with

$$x_1 < x_2 < \cdots < x_n \tag{1.2}$$

are the zeros of $P_n(d\mu, m; x)$ then the Gaussian quadrature formula

$$\int_{\mathbf{R}} f(x) \operatorname{sgn} P_n(d\mu, m; x)^m \, d\mu(x) = \sum_{k=1}^{n} \sum_{j=0}^{m-2} \lambda_{kj} f^{(j)}(x_k) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_{mn-1}$, where the *Christoffel numbers* λ_{kj} are given by

$$\lambda_{kj} = \lambda_{kjn}(d\mu, m) = \int_{\mathbf{R}} A_{kj}(x) \operatorname{sgn} P_n(d\mu, m; x)^m \, d\mu(x) \tag{1.4}$$

and $A_{kj} \in \mathbf{P}_{mn-1}$ are the fundamental polynomials of Hermite interpolation, which satisfy

$$A_{kj}^{(p)}(x_q) = \delta_{kq}\delta_{jp}, \quad j, p = 0, 1, \ldots, m-1, \quad k, q = 1, 2, \ldots, n.$$

As we know, orthogonal polynomials $(m = 2)$ have a long history of study and a classical theory. One of the important contents of this theory are the *Christoffel functions*

$$\lambda_n(d\mu; x) = \min_{P \in \mathbf{P}_{n-1}, P(x)=1} \int_{\mathbf{R}} P(t)^2 \, d\mu(t), \tag{1.5}$$

which are closely related to the *Christoffel numbers*

$$\lambda_{kn}(d\mu) = \lambda_n(d\mu; x_{kn}(d\mu)), \quad k = 1, 2, \ldots, n.$$

Here we accept the notation $P_n(d\mu) = P_n(d\mu, 2)$, $x_{kn}(d\mu) = x_{kn}(d\mu, 2)$, etc. The study and applications of the Christoffel functions can be found in [7].

For the case when $m \in \mathbf{N}_2$ and $j \in \mathbf{M}_2$ the author in [8] defines the Christoffel type functions $\lambda_{jn}(d\mu, m; x)$, which are the extension of $\lambda_n(d\mu; x)$ to the m-orthogonal polynomials, and in [10,11] gives estimations and asymptotics of $\lambda_{jn}(u, m; x)$ for a weight $u \sim W$ for this case, where W is a generalized Jacobi weight. In this paper, we shall establish a complete extension of the Christoffel type functions to the m-orthogonal polynomials for any $m \geqslant 2$ and $j \leqslant m-2$ in this section and further study their properties in Section 2. In Section 3, we shall give the estimations and asymptotics of the Christoffel type functions for some weights including a generalized Jacobi weight.

Given a fixed point $x \in \mathbf{R}$, an index j, $0 \leqslant j \leqslant m-2$, and $n \in \mathbf{N}$, for $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ let the polynomial

$$A_j(P, x; t) = A_{jnm(P,x;t)=\frac{1}{j!}(t-x)^j B_j(P,x;t)P(t)^m} \tag{1.6}$$

with $B_j(P, x; \cdot) \in \mathbf{P}_{m-j-2}$ satisfy the condition

$$A_j^{(i)}(P, x; x) = \delta_{ij}, \quad i = 0, 1, \ldots, m-2. \tag{1.7}$$

It is easy to see that $A_j(P, x; t)$ must exist and be unique.

Definition 1.1. The Christoffel type function $\lambda_{jn}(d\mu, m; x)$ with respect to $d\mu$ is defined by

$$\lambda_{jn}(d\mu, m; x) = \inf_{P \in \mathbf{P}^*_{n-1}(x)} \int_{\mathbf{R}} A_j(P, x; t) \operatorname{sgn}\,[(t-x)P(t)]^m \, d\mu(t) \tag{1.8}$$

for $j \in \mathbf{M}_2$ and by

$$\lambda_{jn}(d\mu, m; x) = \int_{\mathbf{R}} A_j(P, x; t) \operatorname{sgn}\,[(t-x)P(t)]^m \, d\mu(t) \tag{1.9}$$

for $j \in \mathbf{M}_1$, where the polynomial P in (1.9) is the solution of (1.8) in the case when $j \in \mathbf{M}_2$.

Remark 1.1. We shall see that there is a unique polynomial $P \in \mathbf{P}^*_{n-1}(x)$ such that Eq. (1.8) holds for *every* $j \in \mathbf{M}_2$ (Theorem 2.1). So the definition of $\lambda_{jn}(d\mu, m; x)$ for $j \in \mathbf{M}_1$ is reasonable.

Remark 1.2. It is particularly simple to determine $\lambda_{j1}(d\mu, m; x)$: in this case we have that $\mathbf{P}^*_0(x) = \{1\}$, $A_j(P, x; t) = (t-x)^j/j!$, and

$$\lambda_{j1}(d\mu, m; x) = \frac{1}{j!} \int_{\mathbf{R}} (t-x)^j \operatorname{sgn}\,(t-x)^m \, d\mu(t).$$

So in what follows we always assume $n \geqslant 2$.

2. Properties

The expression and the main properties of the polynomial $B_j(P, x; t)$ are as follows.

Lemma 2.1. *We have*

$$B_j(P, x; t) = \sum_{i=0}^{m-j-2} b_i (t-x)^i, \tag{2.1}$$

where

$$b_i = b_i(P, x) = \frac{1}{i!} \left[P(t)^{-m}\right]^{(i)}_{t=x}, \quad i = 0, 1, \ldots . \tag{2.2}$$

Moreover, for $P \in \mathbf{P}^*_{n-1}(x)$ *and* $j \in \mathbf{M}_2$

$$b_{m-j-2} > 0, \quad B_j(P, x; t) > 0, \quad t \in \mathbf{R}. \tag{2.3}$$

Proof. If we rewrite (1.6) in the form

$$A_j(P,x;t)P(t)^{-m} = \frac{1}{j!}(t-x)^j B_j(P,x;t) = \frac{1}{j!}\sum_{i=0}^{m-j-2} b_i(t-x)^{i+j},$$

using (1.7) we can examine (2.2). To prove (2.3) it suffices to apply (2.24) in [9], since $\partial B_j(P,x;\cdot) = m-j-2 \in \mathbf{N}_2$. $\square$

Remark 2.1. By (2.3) we see that the integrand in (1.8) is nonnegative for $j \in \mathbf{M}_2$:

$$A_j(P,x;t)\mathrm{sgn}\,[(t-x)P(t)]^m = |A_j(P,x;t)|.$$

Remark 2.2. For $m \geqslant 3$ the restriction $P \in \mathbf{P}^*_{n-1}(x)$ cannot be replaced by the condition $P \in \mathbf{P}_{n-1}$ with $P(x)=1$ used in the case when $m=2$, for otherwise the function $\lambda_{jn}(d\mu,m;x) = -\infty$ may occur. For example choose $n=3,\ m=4,\ x=0$, and $P(t)=1+dt^2$ $(d>0)$. By a simple calculation we see

$$[P(t)^{-4}]' = -8dt(1+dt^2)^{-5},$$

$$[P(t)^{-4}]'' = -8d(1+dt^2)^{-5} + 80d^2t^2(1+dt^2)^{-6},$$

$$b_0 = 1,\ b_1 = 0,\ b_2 = -4d$$

and

$$A_0(P,0;t) = (1-4dt^2)(1+dt^2)^4.$$

It is easy to see that for any measure $d\mu$

$$\lim_{d\to\infty}\int_{\mathbf{R}} A_0(P,0;t)\,d\mu(t) = -\infty.$$

Remark 2.3. The definition (1.8) is not suitable for $j \in \mathbf{M}_1$. For example for $m \in \mathbf{N}_2,\ j=m-3$, and

$$-2 < x_1 < x_2 < \cdots < x_{n-1} < x = -1,$$

we have

$$A_{m-3}(P,x;t) = [1+b_1(t-x)]P(t)^m, \quad P(t) = \prod_{i=1}^{n-1}\frac{t-x_i}{x-x_i}.$$

Since by Lemma 2.2 in [9]

$$b_1 = a_{1n} = \sum_{i=1}^{n-1}\frac{m}{x_i - x} \leqslant \frac{m}{x_{n-1}-x},$$

we have

$$A_{m-3}(P,x;t) \leqslant \left[1+\frac{m(t-x)}{x_{n-1}-x}\right]P(t)^m = \frac{m(t_0-t)}{x-x_{n-1}}P(t)^m, \quad t \in [-1,1],$$

where

$$t_0 = \frac{x - x_{n-1}}{m} - 1 \in (-1, 1).$$

Then

$$\begin{aligned}
&\int_{-1}^{1} A_{m-3}(P, x; t)\, d\mu(t) \\
&\quad \leqslant \frac{m}{x - x_{n-1}} \left[\int_{-1}^{t_0} (t_0 - t) P(t)^m \, d\mu(t) - \int_{t_0}^{1} (t - t_0) P(t)^m \, d\mu(t) \right] \\
&\quad \leqslant \frac{m}{x - x_{n-1}} \left[P(t_0)^m \int_{-1}^{t_0} (t_0 - t)\, d\mu(t) - P(t_0)^m \int_{t_0}^{1} (t - t_0)\, d\mu(t) \right].
\end{aligned}$$

Thus as $x_{n-1} \to x$ we have $t_0 \to -1$ and hence

$$\lim_{x_{n-1} \to x} \int_{-1}^{1} A_{m-3}(P, x; t)\, d\mu(t) = -\infty.$$

This shows that the definition (1.8) is not suitable for this case.

Let

$$P_\lambda(t) = P(t) + \lambda(t - x)Q(t), \quad P \in \mathbf{P}_{n-1}^*(x), \quad Q \in \mathbf{P}_{n-2}$$

and put $f(\lambda; t) = A_j(P_\lambda, x; t)$ and $g(\lambda; t) = B_j(P_\lambda, x; t)$. The pair (P, Q) is said to satisfy Condition A if there is a number $\delta > 0$ such that the relation $P_\lambda \in \mathbf{P}_{n-1}^*(x)$ holds for every $\lambda \in [0, \delta]$.

Lemma 2.2. *Let $d\mu$ be a measure on $\mathbf{R}$. For a fixed point $x \in \mathbf{R}$ and a fixed index $j \in \mathbf{M}_2$ let a polynomial $P \in \mathbf{P}_{n-1}^*(x)$ satisfy the equation*

$$\int_{\mathbf{R}} A_j(P, x; t) \mathrm{sgn}\, [(t - x)P(t)]^m \, d\mu(t) = \lambda_{jn}(d\mu, m; x). \tag{2.4}$$

If the pair (P, Q) satisfies Condition A, then

$$\int_{\mathbf{R}} [(t - x)P(t)]^{m-1} q(t) \mathrm{sgn}\, [(t - x)P(t)]^m \, d\mu(t) \geqslant 0, \tag{2.5}$$

where

$$q(t) = (t - x)^{j-m+1} [g'_\lambda(0; t)P(t) + m(t - x)g(0; t)Q(t)] \tag{2.6}$$

and

$$q \in \mathbf{P}_{\max\{\partial P - 1, \partial Q\}}. \tag{2.7}$$

Proof. The lemma with $m \in \mathbf{N}_2$ is just Lemma 2 in [8]. By the same argument as that of Lemma 2 in [8] we can prove our lemma with $m \in \mathbf{N}_1$. We omit the details. □

Lemma 2.3. *Let $d\mu$ be a measure on $\mathbf{R}$. Let a point $x \in \mathbf{R}$ be fixed and $j \in \mathbf{M}_2$. Then there exists a polynomial $P \in \mathbf{P}_{n-1}^*(x)$ such that relation* (2.4) *holds.*

Moreover, if a polynomial $P \in \mathbf{P}^*_{n-1}(x)$ *is a solution of* (2.4), *then*

(a) *the zeros of the polynomial P are distinct,*
(b) $\partial P \geqslant n-2$,
(c) *the polynomial P satisfies the orthogonality relation*:

$$\int_{\mathbf{R}} [(t-x)P(t)]^{m-1} q(t) \operatorname{sgn}\,[(t-x)P(t)]^m \, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-2}. \tag{2.8}$$

Proof. There is a gap in the proof of the same lemma with $m \in \mathbf{N}_2$ (Lemma 3) in [8]; so we give a correct and complete proof here.

To prove the first part of the lemma assume that the polynomials $P_N \in \mathbf{P}^*_{n-1}(x)$, $N = 1, 2, \ldots$, satisfy

$$\lim_{N\to\infty} \int_{\mathbf{R}} A_j(P_N, x; t) \operatorname{sgn}\,[(t-x)P_N(t)]^m \, d\mu(t) = \lambda_{jn}(d\mu, m; x).$$

Then (see Remark 2.1)

$$\int_{\mathbf{R}} |A_j(P_N, x; t)| \, d\mu(t) \leqslant c < +\infty, \quad \forall N \in \mathbf{N}.$$

Write

$$A_j(P_N, x; t) = \sum_{k=0}^{mn-2} a_{kN} t^k.$$

By theorem of equivalent norms of finite-dimensional spaces the previous inequalities imply that

$$|a_{kN}| \leqslant c_1 < +\infty, \quad k = 0, 1, \ldots, mn-2, \quad \forall N \in \mathbf{N}.$$

According to Bolzano–Weierstrass theorem, by passing to a subsequence if necessary, we may suppose that $P_N \to P$ $(N \to \infty)$. Then $P \in \mathbf{P}^*_{n-1}(x)$ and relation (2.4) holds.

Let us prove the second part of the lemma.

Assume

$$P(t) = \prod_{k=1}^{r} \left(\frac{t-t_k}{x-t_k}\right)^{p_k},$$

where $-\infty < t_1 < t_2 < \cdots < t_r < +\infty$, $p_1, p_2, \ldots, p_r \in \mathbf{N}$. We claim that

$$\int_{\mathbf{R}} \frac{(t-x)^{m-1} P(t)^m}{t-t_k} \operatorname{sgn}\,[(t-x)P(t)]^m \, d\mu(t) = 0, \quad k = 1, 2, \ldots, r; \tag{2.9}$$

further if $\partial P \leqslant n-2$ then the additional equation

$$\int_{\mathbf{R}} (t-x)^{m-1} P(t)^m \operatorname{sgn}\,[(t-x)P(t)]^m \, d\mu(t) = 0 \tag{2.10}$$

holds.

To prove (2.9) choose $Q(t) = \pm P(t)/(t-t_k)$, $1 \leqslant k \leqslant r$, for which the pair (P, Q) obviously satisfies Condition A. In this case relation (2.6) is of the form

$$q(t) = C(t)Q(t), \tag{2.11}$$

where the function

$$C(t) = (t-x)^{j-m+1}[mg(0;t)(t-x) \pm g'_\lambda(0;t)(t-t_k)]$$

is a polynomial in t, because the functions q, Q, and the function in the bracket are polynomials in t, and $Q(x) \neq 0$. The relation (2.7) shows $\partial q \leqslant \partial Q$ and hence $C(t) \equiv C$. This constant C is not zero, since by (2.3)

$$C = C(t_k) = mg(0;t_k)(t_k - x)^{j-m+2} = mB_j(P, x; t_k)(t_k - x)^{j-m+2} > 0.$$

Substituting q into (2.5), we obtain (2.9).

Similarly, to prove (2.10) choose $Q(t) = \pm P(t)$, for which the pair (P, Q) obviously satisfies Condition A. In this case relation (2.11) holds, where the function

$$C(t) = (t-x)^{j-m+1}[mg(0;t)(t-x) \pm g'_\lambda(0;t)]$$

is also a constant by the same argument as above. To determine this constant we observe that

$$\partial g'_\lambda(0;\cdot) \leqslant m - j - 2 < m - j - 1 = \partial[g(0;\cdot)(\cdot - x)]$$

and hence by (2.3)

$$C(t) \equiv C = \lim_{t\to\infty} C(t) = mb_{m-j-2} > 0.$$

Substituting this q into (2.5), we obtain (2.10).

(a) Suppose to the contrary that $p_k > 1$ holds for some k, $1 \leqslant k \leqslant r$. Choose $Q(t) = -(t - x)P(t)/(t-t_k)^2$, for which the pair (P, Q) obviously satisfies Condition A. In this case by (2.6) we can write q in the form

$$q(t) = C(t)\frac{P(t)}{(t-t_k)^2}, \tag{2.12}$$

where the function

$$C(t) = (t-x)^{j-m+1}[g'_\lambda(0;t)(t-t_k)^2 - mg(0;t)(t-x)^2]$$

is a polynomial in t by the same reason as above. The relation (2.7) shows $\partial q \leqslant \partial Q$ and hence $\partial C \leqslant 1$. Thus we may write the linear function C in the form $C(t) = C_1(t-x) + C_2(t-t_k)$. To determine the sign of the constant C_1 we use (2.3) to get

$$C_1 = \frac{C(t_k)}{t_k - x} = -mg(0;t_k)(t_k - x)^{j-m+2} = -mB_j(P, x; t_k)(t_k - x)^{j-m+2} < 0.$$

Then by (2.12) and (2.9)

$$\begin{aligned}
&\int_{\mathbf{R}} [(t-x)P(t)]^{m-1} q(t) \operatorname{sgn}[(t-x)P(t)]^m \, d\mu(t) \\
&\quad = C_1 \int_{\mathbf{R}} \frac{[(t-x)P(t)]^m}{(t-t_k)^2} \operatorname{sgn}[(t-x)P(t)]^m \, d\mu(t) \\
&\qquad + C_2 \int_{\mathbf{R}} \frac{(t-x)^{m-1}P(t)^m}{t-t_k} \operatorname{sgn}[(t-x)P(t)]^m \, d\mu(t) \\
&\quad = C_1 \int_{\mathbf{R}} \frac{|(t-x)P(t)|^m}{(t-t_k)^2} \, d\mu(t) < 0,
\end{aligned}$$

contradicting (2.5). This proves statement (a).

(b) Suppose not and let $\partial P < n-2$. Choose $Q(t) = -(t-x)P(t)$, which belongs to $\mathbf{P}_{n-2}$ and for which the pair (P, Q) obviously satisfies Condition A. In the present case by (2.6) we get $q(t) = C(t)P(t)$, where

$$C(t) = (t-x)^{j-m+1}[g'_\lambda(0;t) - mg(0;t)(t-x)^2].$$

Again we conclude $\partial C \leqslant 1$ and we may write the linear function C in the form $C(t) = C_1(t-x) + C_2$. To determine the sign of the constant C_1 we observe that

$$\partial g'_\lambda(0;\cdot) \leqslant m-j-2 < m-j = \partial[(\cdot - x)^2 g(0;\cdot)]$$

and hence by (2.3)

$$C_1 = \lim_{t\to\infty} \frac{C(t)}{t-x} = -mb_{m-j-2} < 0.$$

By (2.10) this leads to a contradiction

$$\begin{aligned}
&\int_{\mathbf{R}} [(t-x)P(t)]^{m-1} q(t) \operatorname{sgn}[(t-x)P(t)]^m \, d\mu(t) \\
&\quad = C_1 \int_{\mathbf{R}} |(t-x)P(t)|^m \, d\mu(t) \\
&\qquad + C_2 \int_{\mathbf{R}} (t-x)^{m-1} P(t)^m \operatorname{sgn}[(t-x)P(t)]^m \, d\mu(t) \\
&\quad = C_1 \int_{\mathbf{R}} |(t-x)P(t)|^m \, d\mu(t) < 0
\end{aligned}$$

and proves statement (b).

(c) If $r = n-1$ then relation (2.9) means (2.8), since the set

$$\{P(t)/(t-t_1), \ldots, P(t)/(t-t_{n-1})\}$$

spans the space $\mathbf{P}_{n-2}$; if $r = n-2$ then relations (2.9) and (2.10) imply (2.8), since the set

$$\{P(t)/(t-t_1), \ldots, P(t)/(t-t_{n-2}), P(t)\}$$

again spans the space $\mathbf{P}_{n-2}$. $\square$

Now we can give the following two main results (Theorems 2.1 and 2.2) in this section, which are direct extensions of the corresponding results for orthogonal polynomials.

Theorem 2.1. *Let $d\mu$ be a measure on $\mathbf{R}$ and let $x \in \mathbf{R}$ be fixed.*

(a) *There exists a unique polynomial $P \in \mathbf{P}^*_{n-1}(x)$ such that* (2.4) *holds for every $j \in \mathbf{M}_2$.*
(b) *$\partial P \geqslant n-2$ and the zeros of the polynomial P are distinct.*
(c) *Eq.* (2.4) *is true if and only if the orthogonality relation* (2.8) *holds.*
(d) *We have*

$$\begin{aligned}
\lambda_{m-2,n}(d\mu, m; x) = {}& \min_{Q\in\mathbf{P}_{n-1},\, Q(x)=1} \frac{1}{(m-2)!} \\
&\times \int_{\mathbf{R}} |Q(t)|^m |t-x|^{m-2} \, d\mu(t).
\end{aligned} \tag{2.13}$$

Proof. We begin by showing statement (d). To this end introduce the $(n-1)$-dimensional space

$$G_x = \{(t-x)Q(t) : \ Q \in \mathbf{P}_{n-2}\}. \tag{2.14}$$

Let us consider the extremal problem: to find $P \in \mathbf{P}_{n-1}$ such that $P(x) = 1$ and

$$\int_{\mathbf{R}} |P(t)|^m |t-x|^{m-2}\, d\mu(t) = \min_{Q \in \mathbf{P}_{n-1},\, Q(x)=1} \int_{\mathbf{R}} |Q(t)|^m |t-x|^{m-2}\, d\mu(t). \tag{2.15}$$

It is easy to see that relation (2.15) is true if and only if $R = 1 - P$ $(\in G_x)$ satisfies the equation

$$\int_{\mathbf{R}} |1-R(t)|^m |t-x|^{m-2}\, d\mu(t) = \min_{Q \in G_x} \int_{\mathbf{R}} |1-Q(t)|^m |t-x|^{m-2}\, d\mu(t). \tag{2.16}$$

But this is a problem of L_m approximation to the function 1 with respect to the measure $|t-x|^{m-2}\, d\mu(t)$ from the $(n-1)$-dimentional subspace G_x. By [14, Corollary 2.2, p. 98, Corollary 3.5, p. 111, Theorem 1.11, p. 56] we conclude that there is a *unique* function $R \in G_x$ satisfying (2.16) and further relation (2.16) holds if and only if

$$\int_{\mathbf{R}} [1-R(t)]^{m-1} q(t) |t-x|^{m-2} \operatorname{sgn}\, [1-R(t)]^m\, d\mu(t) = 0, \quad \forall q \in G_x. \tag{2.17}$$

Recalling $R = 1 - P$, relation (2.17) is equivalent to (2.8). This means by (2.14) that there is a *unique* polynomial $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ satisfying (2.15) and further relation (2.15) holds if and only if (2.8) is valid. The orthogonality relation (2.8) shows that the polynomial $(t-x)P(t)$ in t changes sign at least $n-1$ times and hence $P(t)$ changes sign at least $n-2$ times. But $P \in \mathbf{P}_{n-1}$. So P has distinct real zeros only and hence $P \in \mathbf{P}^*_{n-1}(x)$. By (1.6), (2.1), and (2.2) we see

$$A_{m-2}(P, x; t) = \frac{1}{(m-2)!}(t-x)^{m-2} P(t)^m. \tag{2.18}$$

This proves Statements (d).

Meanwhile, we have proved that the solution of the orthogonality relation (2.8) is *unique*. By Lemma 2.3 statements (a)–(c) follow. □

As an immediate consequence of Theorem 2.1 we state

Corollary 2.1. *Let $d\mu$ be a measure on* $\mathbf{R}$ *and let $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ satisfy* (2.8) *or* (2.15). *Then the relation* (1.9) *holds for every $j = 0, 1, \ldots, m-2$.*

Remark 2.5. Corollary 2.1 provides an alternative definition of $\lambda_{jn}(d\mu, m; x)$.

Corollary 2.2. *Let $d\mu$ be a measure on* $\mathbf{R}$. *We have*

$$\lambda_{0n}(d\mu, 2; x) = \lambda_n(d\mu; x). \tag{2.19}$$

Corollary 2.3. *Let $d\mu$ be a measure on* $\mathbf{R}$. *If $P \in \mathbf{P}^*_{n-1}(x)$ satisfies* (2.4) *then the interval $\Delta(d\mu)$ contains at least $n-2$ zeros of P.*

Proof. Suppose to the contrary that the interval $\Delta(d\mu)$ contains $r(\leqslant n-3)$ zeros of P, say, $t_1, \ldots, t_r$. For $q(t) = (t-x)(t-t_1)\cdots(t-t_r)$ we see that the polynomial $[(t-x)P(t)]^{m-1}q(t)$

does not change sign in $\Delta(d\mu)$, which implies that its integral over $\mathbf{R}$ is not zero, contradicting (2.8). □

The second main result in this section is the following

Theorem 2.2. *Let $d\mu$ be a measure on $\mathbf{R}$ and let a polynomial $P \in \mathbf{P}^*_{n-1}(x)$ have the form*

$$P(t) = c \prod_{i=1}^{\partial P} (t - t_i), \quad t_1 < t_2 < \cdots < t_{\partial P}. \tag{2.20}$$

Then the following statements are equivalent:

(a) *The polynomial P satisfies* (2.4) *for* $j \in \mathbf{M}_2$.
(b) *The polynomial P satisfies the orthogonality relation* (2.8).
(c) *We have the Gaussian quadrature formula*

$$\int_{\mathbf{R}} f(t) sgn\, [(t-x)P(t)]^m \, d\mu(t) = \sum_{k=0}^{\partial P} \sum_{j=0}^{m-2} c_{kj} f^{(j)}(t_k) \tag{2.21}$$

exact for all $f \in \mathbf{P}_{(m-1)(\partial P+1)+n-2}$, *where* $t_0 = x$ *and*

$$\lambda_{jn}(d\mu, m; x) = c_{0j}, \quad j = 0, 1, \ldots, m-2. \tag{2.22}$$

(d) *The polynomial P satisfies* (2.15).

Proof. (a) $\Longleftrightarrow$ (b). Use Theorem 2.1(c).

(b) $\Longleftrightarrow$ (c). We separate the cases when $\partial P = n-1$ and $\partial P = n-2$.

Case 1: $\partial P = n-1$. In this case let $A_{kj} \in \mathbf{P}_{mn-2}$ be the fundamental polynomials of Hermite interpolation at the nodes $t_0, t_1, \ldots, t_{n-1}$ with the corresponding multiplicity $m_0 = m-1,\ m_1 = \cdots = m_{n-1} = m$. Then for $f \in \mathbf{P}_{mn-2}$

$$f(t) = \sum_{k=0}^{\partial P} \sum_{j=0}^{m_k - 1} f^{(j)}(t_k) A_{kj}(t). \tag{2.23}$$

Multiplying this with sgn $[(t-x)P(t)]^m$ and then integrating the obtained formula, we have

$$\int_{\mathbf{R}} f(t) \mathrm{sgn}\, [(t-x)P(t)]^m \, d\mu(t) = \sum_{k=0}^{\partial P} \sum_{j=0}^{m_k - 1} c_{kj} f^{(j)}(t_k), \tag{2.24}$$

where

$$c_{kj} = \int_{\mathbf{R}} A_{kj}(t) \mathrm{sgn}\, [(t-x)P(t)]^m \, d\mu(t). \tag{2.25}$$

The formula (2.24) becomes (2.21) if and only if

$$c_{k,m_k-1} = c_{k,m-1} = 0, \quad k = 0, 1, \ldots, \partial P. \tag{2.26}$$

Since

$$A_{k,m-1}(t) = \frac{1}{(m-1)!(t_k - x)^{m-1} P'(t_k)^m} \times [(t-x)P(t)]^{m-1} \frac{P(t)}{t - t_k}, \quad k = 1, 2, \ldots, n-1,$$

relation (2.26) is equivalent to (2.8).

Case 2: $\partial P = n - 2$. In this case we have to consider the generalized Gaussian quadrature formula at the nodes $t_0, t_1, \ldots, t_{n-2}$ with the corresponding multiplicity $m_0 = m_1 = \cdots = m_{n-2} = m$. By Lemma 2.3 in [13] relation (2.21) holds for all $f \in \mathbf{P}_{m(n-1)-1}$ if and only if the orthogonality relation (2.8) is valid.

(b) $\Longleftrightarrow$ (d). Apply Theorem 2.1(c) and (d). □

The relationship between the Christoffel type functions and the Christoffel numbers is given as follows.

Theorem 2.3. *Let $d\mu$ be a measure on $\mathbf{R}$ and for any point $x \in \mathbf{R}$ let $P(x; \cdot) \in \mathbf{P}^*_{n-1}(x)$ satisfy* (2.8). *Let $x_{kn} = x_{kn}(d\mu, m)$.*

(a) *We have*

$$\lambda_{kjn}(d\mu, m) = [sgn\ P_n'(d\mu, m; x_{kn})^m]\lambda_{jn}(d\mu, m; x_{kn}), \quad k = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, m-2. \tag{2.27}$$

(b) *We have*

$$P(x_{kn}; t) = \ell_{kn}(d\mu, m; t) = \frac{P_n(d\mu, m; t)}{P_n'(d\mu, m; x_{kn})(x - x_{kn})}, \quad k = 1, 2, \ldots, n. \tag{2.28}$$

(c) *We have*

$$P(x_{k,n-1}; t) = \ell_{k,n-1}(d\mu, m; t) = \frac{P_{n-1}(d\mu, m; t)}{P_{n-1}'(d\mu, m; x_{k,n-1})(x - x_{k,n-1})}, \quad k = 1, 2, \ldots, n-1. \tag{2.29}$$

(d) *The equality $\partial P(x; \cdot) = n - 2$ is true if and only if equality $x = x_{k,n-1}$ holds for some index $k,\ 1 \leqslant k \leqslant n-1$.*

Proof. (a) Let an index $k,\ 1 \leqslant k \leqslant n$, be fixed. We notice that the polynomial $P_n(d\mu, m)$ satisfies the orthogonality relation by Lemma 2.3 in [13]

$$\int_{\mathbf{R}} P_n(d\mu, m; t)^{m-1} q(t) \mathrm{sgn}\ P_n(d\mu, m; t)^m\, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-1}, \tag{2.30}$$

or equivalently

$$\int_{\mathbf{R}} [(t - x_{kn})P(t)]^{m-1} q(t) \mathrm{sgn}\ [(t - x_{kn})P(t)]^m\, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-1}, \tag{2.31}$$

where $P(t) = \ell_{kn}(d\mu, m; t)$. According to Theorem 2.1 Eq. (2.31) means

$$\lambda_{jn}(d\mu, m; x_{kn}) = \int_{\mathbf{R}} A_j(P, x_{kn}; t) \mathrm{sgn}\ [(t - x_{kn})P(t)]^m\, d\mu(t).$$

Inserting $f(t) = A_j(P, x_{kn}; t)$ into (1.3) and using the above relation, we obtain

$$\begin{aligned}\lambda_{kjn}(d\mu, m) &= \int_{\mathbf{R}} A_j(P, x_{kn}; t)\text{sgn } P_n(d\mu, m; t)^m \, d\mu(t) \\ &= [\text{sgn } P_n'(d\mu, m; x_{kn})^m] \int_{\mathbf{R}} A_j(P, x_{kn}; t)\text{sgn } [(t - x_{kn})P(t)]^m \, d\mu(t) \\ &= [\text{sgn } P_n'(d\mu, m; x_{kn})^m]\lambda_{jn}(d\mu, m; x_{kn}).\end{aligned}$$

This proves (2.27).

(b) The relation (2.28) is a direct consequence of the above conclusion.

(c) To prove (2.29) we use (2.31) with replacing n by $n - 1$

$$\int_{\mathbf{R}} [(t - x_{k,n-1})P(t)]^{m-1} q(t)\text{sgn } [(t - x_{k,n-1})P(t)]^m \, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-2},$$

where $P(t) = \ell_{k,n-1}(d\mu, m; t)$. This by Theorem 2.1 means (2.29).

(d) We observe that by (2.29) equality $\partial P(x; \cdot) = n - 2$ is true if equality $x = x_{k,n-1}$ holds for some index k, $1 \leqslant k \leqslant n - 1$. Conversely, if equality $\partial P(x; \cdot) = n - 2$ is true, then by (2.8) the $(n - 1)$th polynomial $(t - x)P(x; t)$ is the m orthogonal polynomial with respect to $d\mu$ and hence equality $x = x_{k,n-1}$ holds for some index k, $1 \leqslant k \leqslant n - 1$. □

Lemma 2.4. *Let $\partial P = n$, $n-1 \leqslant \partial Q \leqslant n$, $P \neq cQ$, and $p > -1$. Then the following statements are equivalent*:

(a) *the function*

$$f(c, d, p; x) = c|P(x)|^p P(x) + d|Q(x)|^p Q(x) \tag{2.32}$$

has at least $n - 1$ sign changes for every nonzero pair $\{c, d\}$;

(b) *both P and Q have simple real zeros only, and the zeros of P and Q mutually separate each other*.

Proof. We separate the cases when $p = 0$ and $p \neq 0$.

Case 1: $p = 0$.

(a) $\Longrightarrow$ (b). In this case we claim that the polynomial

$$f(c, d, 0; x) = cP(x) + dQ(x)$$

has simple real zeros only for every nonzero pair $\{c, d\}$, because it has at least $n - 1$ sign changes and its degree is n or $n - 1$. We observe that the polynomials P and Q have no common zeros. Suppose not and assume that $P(y) = Q(y) = 0$ for some number $y \in \mathbf{R}$. Then the polynomial

$$f(Q'(y), -P'(y), 0; x) = Q'(y)P(x) - P'(y)Q(x)$$

would have a zero y of multiplicity $\geqslant 2$, a contradiction to the claim.

Further, let x_k, $k = 1, 2, \ldots, n$, in (1.2) be the zeros of P. We claim that each interval (x_k, x_{k+1}), $k = 0, 1, \ldots, n$ $(x_0 = -\infty, x_{n+1} = +\infty)$, cannot contain more than one zero of Q. In fact, suppose to the contrary that the interval (x_j, x_{j+1}), $0 \leqslant j \leqslant n$, contains two zeros of Q. Then we may choose a pair $\{c, d\}$ so that the polynomial $f(c, d, 0; x)$ has a zero y of multiplicity $\geqslant 2$, a contradiction to the claim.

By interchanging P and Q we can prove that each interval (y_k, y_{k+1}), $k = 0, 1, \ldots, \partial Q$, cannot contain more than one zero of P, where y_k, $k = 1, 2, \ldots, \partial Q$, are the zeros of Q

and

$$-\infty = y_0 < y_1 < \cdots < y_{\partial Q} < y_{\partial Q+1} = +\infty.$$

Hence we conclude that between two consecutive zeros of P (or Q) we have exactly one zero of Q (or P).

(b) $\Longrightarrow$ (a). This implication is trivial for the case when $d = 0$. We treat the case when $d \neq 0$. Assume that P has the zeros (1.2). Since

$$f(c, d, 0; x_k) = d|Q(x_k)|^p Q(x_k), \quad k = 1, 2, \ldots, n$$

and

$$Q(x_k)Q(x_{k+1}) < 0, \quad k = 1, 2, \ldots, n-1,$$

the polynomial $f(c, d, 0; x)$ has at least $n-1$ sign changes.

Case 2: $p \neq 0$. In this case we claim that the inequality

$$F(x) = [P(x) + Q(x)][|P(x)|^p P(x) + |Q(x)|^p Q(x)] \geqslant 0$$

holds for every point x. In fact, for a fixed point x, if $|P(x)| > |Q(x)|$ or $P(x) = Q(x)$

$$\operatorname{sgn}[P(x) + Q(x)] = \operatorname{sgn}[|P(x)|^p P(x) + |Q(x)|^p Q(x)] = \operatorname{sgn}[P(x)]$$

and hence $F(x) \geqslant 0$; similarly, if $|P(x)| < |Q(x)|$ or $P(x) = -Q(x)$ then $F(x) \geqslant 0$. This proves our claim.

With the help of this claim if we write

$$\begin{aligned} f(c, d, p; \cdot) = & \left|(\operatorname{sgn} c)|c|^{1/(p+1)} P\right|^p \left[(\operatorname{sgn} c)|c|^{1/(p+1)} P\right] \\ & + \left|(\operatorname{sgn} d)|d|^{1/(p+1)} Q\right|^p \left[(\operatorname{sgn} d)|d|^{1/(p+1)} Q\right] \end{aligned}$$

we can conclude that the function $f(c, d, p; \cdot)$ with $p \neq 0$ has at least $n-1$ sign changes for every nonzero pair $\{c, d\}$ if and only if the polynomial

$$f((\operatorname{sgn} c)|c|^{1/(p+1)}, (\operatorname{sgn} d)|d|^{1/(p+1)}, 0; \cdot)$$

has at least $n-1$ sign changes for every nonzero pair $\{c, d\}$, which is equivalent to that the polynomial $f(c, d, 0; \cdot) = cP + dQ$ has at least $n-1$ sign changes for every nonzero pair $\{c, d\}$. It remains to apply the conclusion of Case 1. $\square$

Theorem 2.4. *Let $d\mu$ be a measure on $\mathbf{R}$ and for any point $x \in \mathbf{R}$ let $P(x; \cdot) \in \mathbf{P}^*_{n-1}(x)$ satisfy* (2.4) *for $j \in \mathbf{M}_2$. If $x \neq y$ then the zeros of the polynomials $(t-x)P(x; t)$ and $(t-y)P(y; t)$ are separate; more precisely, between two consecutive zeros of the polynomial $(t-x)P(x; t)$ (or $(t-y)P(y; t)$) there is precisely one zero of $(t-y)P(y; t)$ (or $(t-x)P(x; t)$).*

In particular, if the polynomial $P(x; \cdot)$ has the form (2.20) *and if $x \notin \{x_{1n}(d\mu, m), \ldots, x_{nn}(d\mu, m)\}$, then each open interval*

$$(x_{kn}(d\mu, m), x_{k+1,n}(d\mu, m)), \quad k = 1, 2, \ldots, n-1,$$

contains precisely one point of $\{x, t_1, \ldots, t_{\partial P}\}$.

Proof. By Theorem 2.1

$$\int_{\mathbf{R}} [(t-x)P(x;t)]^{m-1} q(t) \operatorname{sgn}[(t-x)P(x;t)]^m \, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-2}$$

and

$$\int_{\mathbf{R}} [(t-y)P(y;t)]^{m-1} q(t) \operatorname{sgn}[(t-y)P(y;t)]^m \, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-2}.$$

Then for arbitrary numbers c and d

$$\int_{\mathbf{R}} \{c[(t-x)P(x;t)]^{m-1} \operatorname{sgn}[(t-x)P(x;t)]^m \\ -d[(t-y)P(y;t)]^{m-1} \operatorname{sgn}[(t-y)P(y;t)]^m\} q(t) \, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-2}$$

or equivalently

$$\int_{\mathbf{R}} \{c|(t-x)P(x;t)|^{m-2}(t-x)P(x;t) \\ -d|(t-y)P(y;t)|^{m-2}(t-y)P(y;t)\} q(t) \, d\mu(t) = 0, \quad \forall q \in \mathbf{P}_{n-2}.$$

This shows that the term in the brace has at least $n-1$ sign changes for every nonzero pair $\{c, d\}$. Applying Lemma 2.4 we obtain the first part of the theorem. The second one follows from the first one and Theorem 2.3. $\square$

Theorem 2.5. *Let $d\mu$ be a measure on $\mathbf{R}$ and for any point $x \in \mathbf{R}$ let $P(x;\cdot) \in \mathbf{P}^*_{n-1}(x)$ satisfy* (2.4) *for $j \in \mathbf{M}_2$. Then both $P(x;\cdot)$ and $\lambda_{jn}(d\mu, m; x)$ are continuous with respect to x.*

Proof. To prove our theorem it is convenient to apply Theorem 2.2(d): the inequality

$$\int_{\mathbf{R}} |P(x;t)|^m |t-x|^{m-2} \, d\mu(t) \leqslant \int_{\mathbf{R}} |Q(t)|^m |t-x|^{m-2} \, d\mu(t) \tag{2.33}$$

holds for all $Q \in \mathbf{P}_{n-1}$ with $Q(x) = 1$. Let $x \to \xi$ for some point ξ and let $Q_0 \in \mathbf{P}_{n-1}$ with $Q_0(\xi) = 1$ be arbitrary. Put $Q(t) = Q_0(t - x + \xi)$. Clearly $Q(x) = 1$ and inequality (2.33) holds. Taking $x \to \xi$ in this inequality, suppose, passing to a subsequence if necessary, that $P(x;\cdot) \to P_0$. Then

$$\int_{\mathbf{R}} |P_0(t)|^m |t-\xi|^{m-2} \, d\mu(t) \leqslant \int_{\mathbf{R}} |Q_0(t)|^m |t-\xi|^{m-2} \, d\mu(t).$$

Since $P_0(\xi) = 1$ and $Q_0 \in P_{n-1}$ with $Q_0(\xi) = 1$ is arbitrary, by uniqueness $P_0 = P(\xi;\cdot)$. This prove that $P(x;\cdot)$ is continuous with respect to x. Hence $\lambda_{jn}(d\mu, m; x)$ is also continuous with respect to x. $\square$

Lemma 2.5 (*Shi [12, Theorem 7.4, p. 162]*). *Let $y_1 < y_2 < \cdots < y_n$, $m_k \in \mathbf{N}$, $k = 1, 2, \ldots, n$, and $N = \sum_{k=1}^n m_k - 1$. Denote by $A_{kj} \in \mathbf{P}_N$ the fundamental polynomials of Hermite interpolation at the nodes $y_1, y_2, \ldots, y_n$ with the corresponding multiplicity $m_1, m_2, \ldots, m_n$. If $m_k - j \in \mathbf{N}_1$ and $j < i < m_k$, then*

$$|A_{ki}(x)| \leqslant \frac{j!}{i!} d_k^{i-j} |A_{kj}(x)|, \quad x \in \mathbf{R}, \tag{2.34}$$

where

$$\begin{cases} d_1 = \begin{cases} |y_1 - y_2|, & m_2 > 1, \\ |y_1 - y_3|, & m_2 = 1, \end{cases} \\ d_n = \begin{cases} |y_n - y_{n-1}|, & m_{n-1} > 1, \\ |y_n - y_{n-2}|, & m_{n-1} = 1, \end{cases} \\ d_k = \max\{|y_k - y_{k-1}|, |y_k - y_{k+1}|\}, \quad 2 \leqslant k \leqslant n-1. \end{cases}$$

We formulate two elementary results which are needed later. To this end for each $n \in \mathbf{N}$ let

$$-1 = x_{0n} \leqslant x_{1n} < \cdots < x_{nn} \leqslant x_{n+1,n} = 1,$$

$$x_{kn} = \cos\theta_{kn}, \quad 0 \leqslant \theta_{kn} \leqslant \pi, \quad k = 0, 1, \ldots, n+1,$$

$$d_{kn} = \max\{|x_{kn} - x_{k-1,n}|, |x_{kn} - x_{k+1,n}|\}, \quad k = 1, 2, \ldots, n,$$

and

$$\Delta_n(x) = \frac{(1-x^2)^{1/2}}{n} + \frac{1}{n^2}.$$

Lemma 2.6. *The following statements are equivalent.*

(a) *We have*

$$\theta_{kn} - \theta_{k+1,n} \leqslant \frac{c_1}{n}, \quad k = 0, 1, \ldots, n. \tag{2.35}$$

(b) *We have*

$$x_{k+1,n} - x_{kn} \leqslant c_2 \Delta_n(x_{kn}), \quad k = 0, 1, \ldots, n. \tag{2.36}$$

In addition, the following statements are equivalent.

(c) *We have*

$$\theta_{kn} - \theta_{k+1,n} \geqslant \frac{c_1}{n}, \quad k = 0, 1, \ldots, n. \tag{2.37}$$

(d) *We have*

$$x_{k+1,n} - x_{kn} \geqslant c_2 \Delta_n(x_{kn}), \quad k = 0, 1, \ldots, n. \tag{2.38}$$

Moreover, if inequality (2.35) *is true then*

$$d_{kn} \leqslant c\Delta_n(x_{kn}), \quad k = 1, 2, \ldots, n; \tag{2.39}$$

if inequality (2.37) *is true then*

$$d_{kn} \geqslant c\Delta_n(x_{kn}), \quad k = 1, 2, \ldots, n. \tag{2.40}$$

Proof. Write

$$\begin{aligned} x_{k+1} - x_k &= \cos\theta_{k+1} - \cos\theta_k \\ &= 2\sin\frac{\theta_k - \theta_{k+1}}{2}\sin\frac{\theta_k + \theta_{k+1}}{2}. \end{aligned} \tag{2.41}$$

We have the estimations from above

$$\sin\frac{\theta_k-\theta_{k+1}}{2}\leqslant\frac{\theta_k-\theta_{k+1}}{2}$$

and

$$\begin{aligned}\sin\frac{\theta_k+\theta_{k+1}}{2}&=\sin\left(\theta_k-\frac{\theta_k-\theta_{k+1}}{2}\right)\\&=\sin\theta_k\cos\frac{\theta_k-\theta_{k+1}}{2}-\cos\theta_k\sin\frac{\theta_k-\theta_{k+1}}{2}\\&\leqslant\sin\theta_k+\theta_k-\theta_{k+1}.\end{aligned}$$

Inserting these estimations into (2.41), we obtain

$$x_{k+1}-x_k\leqslant(\theta_k-\theta_{k+1})(\sin\theta_k+\theta_k-\theta_{k+1}). \tag{2.42}$$

On the other hand, we have the estimations from below

$$\sin\frac{\theta_k-\theta_{k+1}}{2}\geqslant\frac{1}{\pi}(\theta_k-\theta_{k+1}),$$

$$\begin{aligned}\sin\frac{\theta_k+\theta_{k+1}}{2}&\geqslant\frac{1}{\pi}(\theta_k+\theta_{k+1})\geqslant\frac{1}{2\pi}(\theta_k+\theta_k-\theta_{k+1})\\&\geqslant\frac{1}{2\pi}\left(\sin\theta_k+\theta_k-\theta_{k+1}\right),\quad\theta_k+\theta_{k+1}\leqslant\pi\end{aligned}$$

and

$$\begin{aligned}\sin\frac{\theta_k+\theta_{k+1}}{2}&\geqslant\frac{1}{\pi}[(\pi-\theta_k)+(\pi-\theta_{k+1})]\geqslant\frac{1}{\pi}[\sin(\pi-\theta_k)+(\theta_k-\theta_{k+1})]\\&=\frac{1}{\pi}\left(\sin\theta_k+\theta_k-\theta_{k+1}\right),\quad\theta_k+\theta_{k+1}>\pi.\end{aligned}$$

Inserting these estimations into (2.41), we obtain

$$x_{k+1}-x_k\geqslant\frac{1}{\pi^2}\left(\theta_k-\theta_{k+1}\right)\left(\sin\theta_k+\theta_k-\theta_{k+1}\right). \tag{2.43}$$

Then the equivalence of statements (a) and (b) as well as of statements (c) and (d) follows from (2.42) and (2.43).

To prove the last part of the lemma by the same arguments as above we can obtain alternative estimations

$$x_k-x_{k-1}\leqslant(\theta_{k-1}-\theta_k)(\sin\theta_k+\theta_{k-1}-\theta_k)$$

and

$$x_k-x_{k-1}\geqslant\frac{1}{\pi^2}\left(\theta_{k-1}-\theta_k\right)\left(\sin\theta_k+\theta_{k-1}-\theta_k\right),$$

from which by (2.35), (2.37), (2.42), and (2.43) we obtain (2.39) and (2.40), respectively. □

Lemma 2.7. *The following statements are equivalent.*

(a) *We have*

$$\frac{c_1}{n} \leqslant \theta_{kn} - \theta_{k+1,n} \leqslant \frac{c_2}{n}, \quad k = 0, 1, \ldots, n. \tag{2.44}$$

(b) *We have*

$$c_3 \Delta_n(x_{kn}) \leqslant x_{k+1,n} - x_{kn} \leqslant c_4 \Delta_n(x_{kn}), \quad k = 0, 1, \ldots, n. \tag{2.45}$$

(c) *We have*

$$\begin{cases} \frac{c_5(1-x_{1n}^2)^{1/2}}{n} \leqslant x_{1n} - x_{0n} \leqslant \frac{c_6(1-x_{1n}^2)^{1/2}}{n}, \\ \frac{c_5(1-x_{kn}^2)^{1/2}}{n} \leqslant x_{k+1,n} - x_{kn} \leqslant \frac{c_6(1-x_{kn}^2)^{1/2}}{n}, \quad k = 1, 2, \ldots, n. \end{cases} \tag{2.46}$$

Proof. (a) $\Longleftrightarrow$ (b) Apply Lemma 2.6.

(b) $\Longleftrightarrow$ (c) We observe that the relation

$$x_1 - x_0 = 1 + x_1 \sim \frac{(1 - x_1^2)^{1/2}}{n} \quad \left(\text{or } x_{n+1} - x_n = 1 - x_n \sim \frac{(1 - x_n^2)^{1/2}}{n}\right)$$

means that $x_1 - x_0 \sim 1/n^2$ (or $x_{n+1} - x_n \sim 1/n^2$). This shows that relation (2.45) with $k = 0$ (or $k = n$) is equivalent to relation (2.46) with $k = 0$ (or $k = n$). Meanwhile for each $k,\ 1 \leqslant k \leqslant n - 1$,

$$\Delta_n(x_k) \sim \frac{(1 - x_k^2)^{1/2}}{n},$$

because $(1 - x_k^2)^{1/2} \geqslant \frac{c}{n}$. This proves the equivalence of statements (b) and (c). $\square$

Theorem 2.6. *Let $d\mu$ be a measure on $\mathbf{R}$. Then for $j \in \mathbf{M}_2$ and $i > j$ we have the estimation*

$$|\lambda_{in}(d\mu, m; x)| \leqslant c d^{i-j} \lambda_{jn}(d\mu, m; x), \tag{2.47}$$

where

$$d = \begin{cases} \max\{|x - x_{k-1,n}|, |x - x_{k+2,n}|\}, & x \in [x_{kn}, x_{k+1,n}],\ 2 \leqslant k \leqslant n - 2, \\ |x - x_{2n}|, & x \leqslant x_{1n}, \\ |x - x_{n-2,n}|, & x \geqslant x_{nn}. \end{cases} \tag{2.48}$$

Further, if $d\mu$ is supported in $[-1, 1]$ *and the condition*

$$\theta_{kn} - \theta_{k+1,n} \leqslant c/n, \quad k = 0, 1, \ldots, n, \tag{2.49}$$

is valid, where $x_{kn} = \cos\,\theta_{kn},\ 0 \leqslant \theta_{kn} \leqslant \pi$, then

$$|\lambda_{in}(d\mu, m; x)| \leqslant c \Delta_n(x)^{i-j} \lambda_{jn}(d\mu, m; x), \\ x \in [-1, 1] \setminus [(x_{1n}, x_{2n}) \cup (x_{n-1,n}, x_{nn})]. \tag{2.50}$$

Proof. Let $P \in \mathbf{P}_{n-1}^*(x)$ satisfy (2.4) for $j \in \mathbf{M}_2$, where $x \in [x_{kn}, x_{k+1,n}],\ 2 \leqslant k \leqslant n - 2$. By Theorem 2.4 the interval $[x_{k-1}, x_k)$ must contain a zero of P, say, y, and the interval $(x_{k+1}, x_{k+2}]$ must contain a zero of P, say, z. Then by (1.8), (1.9), and (2.34)

$$|\lambda_{in}(d\mu, m; x)| \leqslant c(\max\{|x - y|, |x - z|\})^{i-j} \lambda_{jn}(d\mu, m; x) \leqslant c d^{i-j} \lambda_{jn}(d\mu, m; x).$$

If $x \leqslant x_1$ then by Theorem 2.4 the interval $(-\infty, x)$ contains no zero of P and the interval $(x_1, x_2]$ must contain a zero of P, say, y. Then by (1.8), (1.9), and (2.34)

$$|\lambda_{in}(d\mu, m; x)| \leqslant c|x - y|^{i-j}\lambda_{jn}(d\mu, m; x) \leqslant cd^{i-j}\lambda_{jn}(d\mu, m; x).$$

Similarly, if $x \geqslant x_n$ then we can obtain (2.47).

Relation (2.50) follows from (2.48) and (2.36). □

Remark 2.6. It is difficult to give the estimation of $\lambda_{in}(d\mu, m; x)$ for $x \in (x_{1n}, x_{2n}) \cup (x_{n-1,n}, x_{nn})$, because in this case the number d may be large enough.

We state a useful result which is needed in the next section.

Theorem 2.7 (*Shi [8, Theorem 3]*). *Let $d\mu$ be a measure on* $\mathbf{R}$. *If $m \in \mathbf{N}_2$ then*

$$\lambda_{0n}(d\mu, m; x) \geqslant \lambda_{mn/2}(d\mu; x). \tag{2.51}$$

3. Estimations and asymptotics

Lemma 3.1 (*Nevai [6, Lemma 6.3.8, p. 108]*). *Let $v(x) = (1 - x^2)^{-1/2}$ and*

$$K_n(v, x; t) = \frac{T_n(x)T_{n-1}(t) - T_{n-1}(x)T_n(t)}{\pi(x - t)}, \quad n \geqslant 2. \tag{3.1}$$

Then

$$|K_n(v, x; t)| \leqslant c \min\left\{n, \frac{(1 - x^2)^{1/2} + (1 - t^2)^{1/2}}{|x - t|}\right\}, \quad x, t \in [-1, 1], \tag{3.2}$$

where c is an absolute constant.

Lemma 3.2 (*Freud [4, (3.7), p. 102; 104]*). *Let*

$$\pi_{n-1}(x; t) = \frac{K_n(v, x; t)}{K_n(v, x; x)}.$$

Then

$$|\pi_{n-1}(x; t)| \leqslant 4, \quad n \geqslant 3, \quad x, t \in [-1, 1], \tag{3.3}$$

and

$$K_n(v, x; x) \sim n, \quad |x| \leqslant 1. \tag{3.4}$$

By [4, Theorem 3.1, p. 19] the polynomial $\pi_{n-1}(x; t)$ in t has simple real zeros only and hence

$$\pi_{n-1}(x; \cdot) \in \mathbf{P}^*_{n-1}(x). \tag{3.5}$$

Lemma 3.3 (*Shi [10, Lemma 6]*). *We have*

$$|b_i(\pi_{n-1}(x; \cdot); x)| \leqslant c\Delta_n(x)^{-i}, \quad |x| \leqslant 1, \quad i = 0, 1, \ldots . \tag{3.6}$$

Theorem 3.1. *Let $d\mu$ be a measure supported on $[-1, 1]$ and satisfy*

$$\mu \in \mathrm{Lip}_M \gamma, \quad 0 < \gamma \leqslant 1. \tag{3.7}$$

Then for $j \in \mathbf{M}_2$

$$\lambda_{jn}(d\mu, m; x) \leqslant cn^{2-m-\gamma} \Delta_n(x)^{2+j-m}. \tag{3.8}$$

Proof. Choose

$$P(x; t) = \pi_{n-1}(x; t).$$

Using the definition of $\lambda_{jn}(d\mu, m; x)$ and applying (1.6) and (2.1), we obtain

$$\begin{aligned}
&\lambda_{jn}(d\mu, m; x) \\
&\quad \leqslant \int_{-1}^{1} |A_j(\pi_{n-1}, x; t)| \, d\mu(t) \\
&\quad = \int_{-1}^{1} \left| \sum_{i=0}^{m-j-2} b_i(\pi_{n-1}, x; t)(t-x)^{j+i} \pi_{n-1}(x; t)^m \right| d\mu(t).
\end{aligned}$$

By (3.6), (3.2), and (3.4)

$$\begin{aligned}
&\lambda_{jn}(d\mu, m; x) \\
&\quad \leqslant c \sum_{i=0}^{m-j-2} \Delta_n(x)^{-i} \int_{-1}^{1} |(t-x)^{j+i} \pi_{n-1}(x; t)^m| \, d\mu(t) \\
&\quad = c \sum_{i=0}^{m-j-2} \Delta_n(x)^{-i} \int_{-1}^{1} \left| \frac{(t-x)K_n(v, x; t)}{K_n(v, x; x)} \right|^{j+i} |\pi_{n-1}(x; t)|^{m-j-i} \, d\mu(t) \\
&\quad \leqslant c \sum_{i=0}^{m-j-2} n^{-j-i} \Delta_n(x)^{-i} \int_{-1}^{1} |\pi_{n-1}(x; t)|^{m-j-i} \, d\mu(t).
\end{aligned} \tag{3.9}$$

We have to estimate the integral

$$\sigma_q = \int_{-1}^{1} |\pi_{n-1}(x; t)|^q \, d\mu(t), \quad q \geqslant 2.$$

We divide the integral σ_q into the three parts:

$$\begin{aligned}
\sigma_q &= \int_{x-1/n}^{x+1/n} |\pi_{n-1}(x; t)|^q \, d\mu(t) + \int_{-1}^{x-1/n} |\pi_{n-1}(x; t)|^q \, d\mu(t) \\
&\quad + \int_{x+1/n}^{1} |\pi_{n-1}(x; t)|^q \, d\mu(t) \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

It is simple to estimate S_1, since by (3.3) and (3.7)

$$S_1 \leqslant 2^{2q+1} M n^{-\gamma}.$$

If $1+x \leqslant 1/n$ then $S_2 = 0$; otherwise applying (3.2), (3.4), and (3.7), and using partial integration, we obtain

$$\begin{aligned}
S_2 &\leqslant cn^{-q} \int_{-1}^{x-1/n} (x-t)^{-q}\, d\mu(t) \\
&= cn^{-q} \int_{-1}^{x-1/n} (x-t)^{-q} d[\mu(t)-\mu(x)] \\
&\leqslant cn^{-q} \left\{ \frac{\mu(x)-\mu(-1)}{(1+x)^q} + qM \int_{-1}^{x-1/n} (x-t)^{\gamma-q-1}\, dt \right\} \\
&\leqslant cMn^{-q} \left\{ (1+x)^{\gamma-q} + q(q-\gamma)^{-1} n^{q-\gamma} \right\} \\
&\leqslant cMn^{-\gamma}.
\end{aligned}$$

Similarly, if $1-x \leqslant 1/n$ then $S_3 = 0$; otherwise

$$S_3 \leqslant cMn^{-\gamma}.$$

Thus $\sigma_q \leqslant cn^{-\gamma}$ and by (3.9) we obtain (3.8). $\square$

Theorem 3.2. *Let $d\mu$ be a measure supported on $[-1,1]$ and $j \in \mathbf{M}_2$. Then*

$$\limsup_{n\to\infty} n^{j+1} \lambda_{jn}(d\mu, m; x) \leqslant c\mu'(x)(1-x^2)^{(j+1)/2} \tag{3.10}$$

holds for almost every $x \in [-1,1]$.

Proof. By (3.9) and (3.2)–(3.4)

$$\begin{aligned}
&\lambda_{jn}(d\mu, m; x) \\
&\quad\leqslant c \sum_{i=0}^{m-j-2} \Delta_n(x)^{-i} \int_{-1}^{1} |(t-x)^{j+i} \pi_{n-1}(x;t)^m|\, d\mu(t) \\
&\quad\leqslant c \sum_{i=0}^{m-j-2} \Delta_n(x)^{-i} \int_{-1}^{1} \left| (t-x)^{j+i} \left[\frac{K_n(v,x;t)}{K_n(v,x;x)} \right]^{i+j+2} \right| d\mu(t) \\
&\quad\leqslant c \sum_{i=0}^{m-j-2} n^{-i-j-1} \Delta_n(x)^{-i} \\
&\qquad\times \lambda_n(v;x) \int_{-1}^{1} |(t-x)K_n(v,x;t)|^{i+j} K_n(v,x;t)^2\, d\mu(t) \\
&\quad\leqslant cn^{-j-1} \sum_{i=0}^{m-j-2} \left[n\Delta_n(x) \right]^{-i} \\
&\qquad\times \lambda_n(v;x) \int_{-1}^{1} \left[(1-x^2)^{1/2} + (1-t^2)^{1/2} \right]^{i+j} K_n(v,x;t)^2\, d\mu(t)
\end{aligned}$$

$$= cn^{-j-1} \sum_{i=0}^{m-j-2} \left[n\Delta_n(x)\right]^{-i} \sum_{s=0}^{i+j} \binom{i+j}{s} (1-x^2)^{s/2}$$
$$\times \lambda_n(v;x) \int_{-1}^{1} K_n(v,x;t)^2 (1-t^2)^{(i+j-s)/2}\, d\mu(t)$$

or equivalently

$$n^{j+1}\lambda_{jn}(d\mu,m;x) \leqslant c \sum_{i=0}^{m-j-2} \left[n\Delta_n(x)\right]^{-i} \sum_{s=0}^{i+j} \binom{i+j}{s} (1-x^2)^{s/2}$$
$$\times \lambda_n(v;x) \int_{-1}^{1} K_n(v,x;t)^2 (1-t^2)^{(i+j-s)/2}\, d\mu(t).$$

Here we need a formula given by Nevai in [6, Lemma 6.2.32, p. 93]

$$\lim_{n\to\infty} \lambda_n(v;x) \int_{-1}^{1} K_n(v,x;t)^2\, d\mu(t) = \mu'(x)(1-x^2)^{1/2},$$

which holds for almost every $x \in [-1,1]$. Hence for almost every $x \in [-1,1]$

$$\limsup_{n\to\infty} n^{j+1}\lambda_{jn}(d\mu,m;x)$$
$$\leqslant c \sum_{i=0}^{m-j-2} (1-x^2)^{-i/2} \sum_{s=0}^{i+j} \binom{i+j}{s} (1-x^2)^{s/2}(1-x^2)^{(i+j-s)/2}$$
$$\times \mu'(x)(1-x^2)^{1/2}$$
$$\leqslant c\mu'(x)(1-x^2)^{(j+1)/2}. \qquad \square$$

Lemma 3.4 (*Ullman [16, pp. 471–472]*). *Let $d\mu$ be a measure supported on* $[-1,1]$. *Then*

$$\lim_{n\to\infty} \gamma_n(d\mu)^{1/n} = 2 \tag{3.11}$$

if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\mu)) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}} \tag{3.12}$$

holds for every $f \in C[-1,1]$.

Lemma 3.5 (*Nevai [6, Lemma 5.1, p. 49]*). *Let $d\mu$ be a measure with a compact support and let f be continuous on $\Delta(d\mu)$ with the modulus of continuity ω. Then*

$$\left|\sum_{k=1}^{n} f(x_{kn}) - \sum_{k=0}^{n-1} \int_{\mathbf{R}} f(x) P_k(d\mu;x)^2\, d\mu(x)\right|$$
$$\leqslant n\omega(n^{-1/3}) \left[1 + \frac{1}{2}|\Delta(d\mu)|^3\right] \tag{3.13}$$

holds for $n > |\Delta(d\mu)|^{-3}$, where $|\Delta(d\mu)|$ denotes the length of the interval $\Delta(d\mu)$.

Lemma 3.6. *Let $d\mu$ be a measure supported on* $[-1, 1]$ *and let relation* (3.11) *prevail. Then*

$$\limsup_{n\to\infty} n\lambda_n(d\mu; x) = \pi\mu'(x)(1-x^2)^{1/2} \tag{3.14}$$

holds for almost every $x \in [-1, 1]$.

Proof. Using the formula

$$\lambda_n(d\mu; x) = \left[\sum_{k=0}^{n-1} P_k(d\mu; x)^2\right]^{-1} \tag{3.15}$$

the inequality (3.13) becomes

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(x_{kn}) - \sum_{k=0}^{n-1}\int_{\mathbf{R}} \frac{f(x)}{n\lambda_n(d\mu; x)}\, d\mu(x)\right| \leqslant 5\omega(n^{-1/3}),$$

which, together with (3.12), implies that the relation

$$\lim_{n\to\infty}\int_{-1}^{1} \frac{f(x)}{n\lambda_n(d\mu; x)}\, dx = \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)\,dx}{\sqrt{1-x^2}} \tag{3.16}$$

holds for every $f \in C[-1, 1]$. Using one-sided approximation we conclude that relation (3.16) remains true if f is the characteristic function of an interval. Then by the same argument as that of Theorem 6.2.54 in [6, pp. 104–105] we obtain that relation

$$\limsup_{n\to\infty} n\lambda_n(d\mu; x) \geqslant \pi\mu'(x)(1-x^2)^{1/2}$$

holds for almost every $x \in [-1, 1]$. By (3.10) we obtain (3.14). □

Remark 3.1. Lemma 3.6 improves Theorems 6.2.54 and 6.2.55 in [6, pp. 104–105], there relation (3.14) is proved for a measure $d\mu$ satisfying $\lim_{n\to\infty}\gamma_{n+1}(d\mu)/\gamma_n(d\mu) = 2$ and $\mu'(x) > 0$, a.e., respectively.

Corollary 3.1. *Let $d\mu$ be a measure supported on* $[-1, 1]$ *and let relation* (3.11) *prevail. If $m \in \mathbf{N}_2$ then for almost every $x \in [-1, 1]$*

$$\frac{2\pi}{m}\mu'(x)(1-x^2)^{1/2} \leqslant \limsup_{n\to\infty} n\lambda_{0n}(d\mu, m; x) \leqslant c\mu'(x)(1-x^2)^{1/2}.$$

Proof. This follows from (2.51), (3.14), and (3.10). □

In what follows we shall give an estimation of $\lambda_{jn}(u, m; x)$ for a weight

$$u \sim W, \quad a.e., \tag{3.17}$$

where W is a *generalized Jacobi weight*:

$$W(x) = \prod_{i=1}^{r} |x-t_i|^{p_i}, \quad |x| < 1, \quad W(x) = 0, \quad |x| \geqslant 1,$$
$$-1 = t_1 < t_2 < \cdots < t_r = 1 \ (r \geqslant 2), \quad p_i > -1, \ i = 1, 2, \ldots, r. \tag{3.18}$$

Theorem 3.3. *Let relation* (3.17) *prevail. Then with the constants associated with the symbol* $\sim$ *depending on u and m,*

$$\lambda_{jn}(u,m;x) \sim \lambda_n(u;x)\Delta_n(x)^j \sim \frac{1}{n}W_n(x)\Delta_n(x)^j, \quad x\in[-1,1], \quad j\in \mathbf{M}_2. \tag{3.19}$$

Here

$$W_n(x) = \left[(1+x)^{1/2}+\frac{1}{n}\right]^{2p_1+1}\left[(1-x)^{1/2}+\frac{1}{n}\right]^{2p_r+1} \times\prod_{i=2}^{r-1}\left[|x-t_i|+\frac{1}{n}\right]^{p_i}. \tag{3.20}$$

Proof. The relation (3.19) may be proved by the same argument as that of Theorem 1 in [11], there this argument is applied to the case when $j\in\mathbf{M}_2$ with $m\in\mathbf{N}_2$ and remains valid for $j\in\mathbf{M}_2$ with $m\geqslant 2$. $\square$

Lemma 3.7 (*Shi [10, Lemma 8]*). *Let* $x_{kn}=x_{kn}(d\mu,m)$. *Then*

$$x_{1n}<x_{1,n-1}<x_{2n}<x_{2,n-1}<\cdots<x_{n-1,n}<x_{n-1,n-1}<x_{nn}.$$

Lemma 3.8 (*Hardy [5, Theorem 27, pp. 71–72]*). *Let* $A, B, p\geqslant 0$ *and* $AB+p>0$. *Then*

$$(A+B)^p\leqslant c(p)(A^p+B^p). \tag{3.21}$$

Lemma 3.9. *Let* $b>a>0$ *and* $d=(b-a)/h>2$. *Then*

$$(b-h)^p-(a+h)^p\geqslant c(b^p-a^p), \tag{3.22}$$

where

$$c=\begin{cases}1, & p\leqslant 0,\\ \frac{(d-1)^p-1}{d}, & 0<p<1,\\ \frac{(d-2)(d-1)^p}{d^p}, & p\geqslant 1.\end{cases} \tag{3.23}$$

Proof. Rewrite (3.22) in the form with $\delta=h/a$

$$[1+(d-1)\delta]^p-(1+\delta)^p\geqslant c[(1+d\delta)^p-1]. \tag{3.24}$$

For $p\leqslant 0$ we have $[1+(d-1)\delta]^p\geqslant(1+d\delta)^p$ and $(1+\delta)^p\leqslant 1$. Hence (3.24) with $c=1$ is valid.

For $p>0$ we consider the problem to minimize the function

$$g(\delta,c)=c \tag{3.25}$$

subject to the condition

$$g_1(\delta,c)=[1+(d-1)\delta]^p-(1+\delta)^p-c[(1+d\delta)^p-1]\geqslant 0. \tag{3.26}$$

If a pair $\{\delta,c\}$ is a solution of this problem then according to Theorem 3.4 in [1] there is a pair $\{\lambda_0,\lambda_1\}$, $\lambda_0,\lambda_1\geqslant 0$, $\lambda_0+\lambda_1>0$, such that

$$-\lambda_1\frac{\partial g_1}{\partial\delta}=0, \tag{3.27}$$

$$\lambda_0 - \lambda_1 \frac{\partial g_1}{\partial c} = 0 \tag{3.28}$$

and

$$\lambda_1 g_1 = 0. \tag{3.29}$$

Eq. (3.28) shows that $\lambda_1 > 0$, for otherwise it would lead to $\lambda_0 = \lambda_1 = 0$, a contradiction. Thus Eq. (3.27) yields

$$\frac{\partial g_1}{\partial \delta} = 0,$$

that is,

$$(d-1)[1+(d-1)\delta]^{p-1} - (1+\delta)^{p-1} - cd[(1+d\delta)]^{p-1} = 0.$$

This gives

$$c = \frac{(d-1)[1+(d-1)\delta]^{p-1} - (1+\delta)^{p-1}}{d[(1+d\delta)]^{p-1}}.$$

If $0 < p < 1$ then

$$\begin{aligned} c &\geqslant \frac{(d-1)[(d-1)+(d-1)\delta]^{p-1} - (1+\delta)^{p-1}}{d[(1+d\delta)]^{p-1}} \\ &= \frac{(d-1)^p - 1}{d}\left(\frac{1+\delta}{1+d\delta}\right)^{p-1} \geqslant \frac{(d-1)^p - 1}{d}; \end{aligned}$$

if $p \geqslant 1$ then

$$c \geqslant \frac{d-2}{d}\left(\frac{1+(d-1)\delta}{1+d\delta}\right)^{p-1} \geqslant \frac{d-2}{d}\left(\frac{d-1}{d}\right)^{p-1} = \frac{(d-2)(d-1)^{p-1}}{d^p}.$$

□

Lemma 3.10. *Let* $-1 \leqslant a < b \leqslant 1$ *and* $d = (b-a)/h > 4$. *Let* (3.17) *prevail. Then*

$$\int_{a+h}^{b-h} u(t)\,dt \geqslant c(u,d) \int_a^b u(t)\,dt. \tag{3.30}$$

Proof. Put

$$\delta = \frac{d-2}{2d} \min_{1 \leqslant i \leqslant r-1} (t_{i+1} - t_i) \tag{3.31}$$

and

$$c_1 = \left[\int_{-1}^{1} u(t)\,dt\right]^{-1} \inf_{-1 \leqslant \tau \leqslant 1-\delta} \int_\tau^{\tau+\delta} u(t)\,dt > 0. \tag{3.32}$$

We separate the cases when $b - a - 2h \geqslant \delta$ and $b - a - 2h < \delta$.

Case 1: $b-a-2h \geqslant \delta$. In this case by (3.32)

$$\int_{a+h}^{b-h} u(t)\,dt \geqslant \inf_{-1\leqslant\tau\leqslant 1-\delta}\int_{\tau}^{\tau+\delta} u(t)\,dt$$
$$\geqslant c_1\int_{-1}^{1} u(t)\,dt \geqslant c_1\int_a^b u(t)\,dt.$$

Case 2: $b-a-2h < \delta$. In this case since

$$b-a-2h = b-a-\frac{2(b-a)}{d} = \frac{(d-2)(b-a)}{d},$$

by (3.31)

$$b-a < \frac{1}{2}\min_{1\leqslant i\leqslant r-1}(t_{i+1}-t_i).$$

This shows that the interval $[a,b]$ contains at most one point of t_i's. Assume that the index $i,\ 1\leqslant i\leqslant r$, satisfies

$$\min_{t\in[a,b]}|t-t_i| = \min_{1\leqslant j\leqslant r}\min_{t\in[a,b]}|t-t_j|.$$

Again we separate the cases when $t_i \in [a,b]$ and $t_i \notin [a,b]$.

Case 2.1: $t_i \in [a,b]$. By calculation

$$\int_{a+h}^{b-h} u(t)\,dt \geqslant c\int_{a+h}^{b-h} |t-t_i|^{p_i}\,dt$$
$$= \begin{cases} \frac{c}{p_i+1}\left[(b-t_i-h)^{p_i+1}+(t_i-a-h)^{p_i+1}\right], & t_i\in(a+h,b-h),\\ \frac{c}{p_i+1}\left[(b-t_i-h)^{p_i+1}-(a-t_i+h)^{p_i+1}\right], & t_i\leqslant a+h,\\ \frac{c}{p_i+1}\left[(t_i-a-h)^{p_i+1}-(t_i-b+h)^{p_i+1}\right], & t_i\geqslant b-h.\end{cases}$$

On the other hand, we have

$$\int_a^b u(t)\,dt \leqslant c\int_a^b |t-t_i|^{p_i}\,dt = \frac{c}{p_i+1}\left[(b-t_i)^{p_i+1}+(t_i-a)^{p_i+1}\right].$$

For $t_i \in (a+h,b-h)$, using inequality (3.21), we see

$$(b-t_i-h)^{p_i+1}+(t_i-a-h)^{p_i+1} \geqslant \left(\frac{b-a-2h}{2}\right)^{p_i+1}$$
$$= \left[\left(\frac{d-2}{2d}\right)(b-a)\right]^{p_i+1} \geqslant \frac{1}{2}\left(\frac{d-2}{2d}\right)^{p_i+1}\left[(b-t_i)^{p_i+1}+(t_i-a)^{p_i+1}\right].$$

For $(a\leqslant)t_i\leqslant a+h$, we have

$$\frac{1}{h}\left[(b-t_i)-(t_i-a)\right] \geqslant \frac{1}{h}\left[(b-a-h)-h\right] = d-2 > 2,$$

$$b-t_i \geqslant b-a-h = (d-1)h \geqslant (d-1)(t_i-a),$$

and hence by (3.22)

$$
\begin{aligned}
&(b-t_i-h)^{p_i+1}-(a-t_i+h)^{p_i+1}\\
&\quad \geqslant (b-t_i-h)^{p_i+1}-(t_i-a+h)^{p_i+1}\\
&\quad \geqslant c\left[(b-t_i)^{p_i+1}-(t_i-a)^{p_i+1}\right]\\
&\quad \geqslant \frac{c[(d-1)^{p_i+1}-1]}{(d-1)^{p_i+1}+1}\left[(b-t_i)^{p_i+1}+(t_i-a)^{p_i+1}\right].
\end{aligned}
$$

Similarly, if $t_i \geqslant b-h$, then

$$
\begin{aligned}
&(t_i-a-h)^{p_i+1}-(t_i-b+h)^{p_i+1}\\
&\quad \geqslant \frac{c[(d-1)^{p_i+1}-1]}{(d-1)^{p_i+1}+1}\left[(b-t_i)^{p_i+1}+(t_i-a)^{p_i+1}\right].
\end{aligned}
$$

Thus in all the cases inequality (3.30) follows.

Case 2.2: $t_i \notin [a,b]$. Suppose without loss of generality that $t_i < a$. Then

$$
\begin{aligned}
&\int_{a+h}^{b-h} u(t)\,dt \geqslant c\int_{a+h}^{b-h} |t-t_i|^{p_i}\,dt\\
&\quad = \frac{c}{p_i+1}\left[(b-t_i-h)^{p_i+1}-(a-t_i+h)^{p_i+1}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{a}^{b} u(t)\,dt \leqslant c\int_{a}^{b} |t-t_i|^{p_i}\,dt\\
&\quad = \frac{c}{p_i+1}\left[(b-t_i)^{p_i+1}-(a-t_i)^{p_i+1}\right].
\end{aligned}
$$

Applying (3.22) we get (3.30). □

Lemma 3.11. *If* $p \geqslant 0,\ B > A \geqslant 0$, *and* $\sigma = \pm 1$, *then*

$$\frac{(B+\sigma A)(B^p+A^p)}{B^{p+1}+\sigma A^{p+1}} \leqslant 2. \tag{3.33}$$

Proof. The inequality (3.33) with $\sigma=-1$ may be found in [10, Lemma 5]; the one with $\sigma=1$ may be proved similarly. □

Lemma 3.12. *Let* $p \geqslant 0,\ B_n \geqslant A_n \geqslant 0,\ \sigma = \pm 1$, *and*

$$B_n^{p+1}+\sigma A_n^{p+1} \leqslant \frac{C}{n}\left[\left(B_n+\frac{1}{n}\right)^p+\left(A_n+\frac{1}{n}\right)^p\right]. \tag{3.34}$$

Then

$$B_n+\sigma A_n \leqslant \frac{c(C,p)}{n}. \tag{3.35}$$

Proof. If $A_n \leqslant 1/n$ then by (3.34) and (3.21)

$$B_n^{p+1} \leqslant \frac{A_n^p}{n} + \frac{C}{n}\left[\left(B_n + \frac{1}{n}\right)^p + \left(A_n + \frac{1}{n}\right)^p\right]$$
$$\leqslant \frac{2C+1}{n}\left(B_n + \frac{1}{n}\right)^p \leqslant \frac{(2C+1)2^p}{n}\left[B_n^p + \left(\frac{1}{n}\right)^p\right]$$

and hence

$$B_n + \sigma A_n \leqslant 2B_n \leqslant \frac{c}{n}.$$

If $A_n > 1/n$ then by (3.34)

$$B_n^{p+1} + \sigma A_n^{p+1} \leqslant \frac{c}{n}\left(B_n^p + A_n^p\right)$$

and hence by (3.33)

$$B_n + \sigma A_n \leqslant \frac{c}{n} \cdot \frac{(B_n + \sigma A_n)(B_n^p + A_n^p)}{B_n^{p+1} + \sigma A_n^{p+1}} \leqslant \frac{c}{n}. \qquad \square$$

Remark 3.1. Unfortunately, Lemma 3.12 is not true in general for the case when $-1 < p < 0$, although we need such a result later. For example, let $A_n = n^{-3/4}$, $\varepsilon_n = n^{-1/4}$, and $B_n = (1+\varepsilon_n)^2 A_n$. Then

$$B_n^{1/2} - A_n^{1/2} = \varepsilon_n A_n^{1/2} = \varepsilon_n(1+\varepsilon_n)A_n B_n^{-1/2}$$
$$\leqslant 2n^{-1}B_n^{-1/2} \leqslant n^{-1}(B_n^{-1/2} + A_n^{-1/2})$$
$$\leqslant 2n^{-1}\left[\left(B_n + \frac{1}{n}\right)^{-1/2} + \left(A_n + \frac{1}{n}\right)^{-1/2}\right],$$

which shows that inequality (3.34) is true. But inequality (3.35) is violated, because

$$B_n - A_n = 2\varepsilon_n A_n + \varepsilon_n^2 \geqslant n^{-1/2}.$$

Lemma 3.13 (*Shi [11, Lemma 2]*). *Let* $P \in \mathbf{P}_n$. *Then*

$$\max_{|x|\leqslant 1} \left|P'(x)W_n(x)\delta_n(x)\right| \leqslant cn \max_{|x|\leqslant 1} |P(x)W_n(x)|. \tag{3.36}$$

Moreover, if (3.17) *is true then*

$$\max_{|x|\leqslant 1} |P(x)W_n(x)| \leqslant cn \int_{-1}^{1} |P(x)|u(x)\,dx \tag{3.37}$$

and

$$\int_{-1}^{1} |P'(x)|(1-x^2)^{1/2}u(x)\,dx \leqslant cn \int_{-1}^{1} |P(x)|u(x)\,dx, \tag{3.38}$$

where $\delta_n(x) = (1-x^2)^{1/2} + n^{-1}$ *and c is a constant independent of n and P.*

Lemma 3.14 (*Shi [11, Lemma 3]*). *Let* $y_n = \cos\alpha_n$ *and* $z_n = \cos\beta_n$. *If* $|\alpha_n - \beta_n| \leqslant C/n$ *then with the constants associated with the symbol* $\sim$ *depending on w and C only*

$$W_n(y_n) \sim W_n(z_n). \tag{3.39}$$

Theorem 3.4. *Let* $x_{kn} = x_{kn}(u, m)$ *and let relation* (3.17) *with*

$$p_i \geqslant 0, \quad i = 2, 3, \ldots, r-1, \tag{3.40}$$

prevail. Then the relation (2.44) *is valid.*

Proof. The proof follows and properly modifies the ideas of Nevai in [6, pp. 164–167]. Meanwhile according to Lemma 2.7 it is enough to prove (2.45). We use the notation $\ell_j = \ell_{jn}(u, m)$ of (2.28) and break the proof into two claims.

Claim 1.

$$x_{k+1} - x_k \leqslant c\Delta_n(x_k), \quad k = 0, 1, \ldots, n.$$

Choose n_0 so large that for $n \geqslant n_0$

$$\max_{0 \leqslant k \leqslant n} (x_{k+1,n} - x_{kn}) \leqslant \frac{1}{2} \min_{1 \leqslant j \leqslant r-1} (t_{j+1} - t_j).$$

For a fixed index $k,\ 0 \leqslant k \leqslant n$, assume that an index $i,\ 1 \leqslant i \leqslant r$, satisfies

$$\min_{t \in [x_k, x_{k+1}]} |t - t_i| = \min_{1 \leqslant j \leqslant r} \min_{t \in [x_k, x_{k+1}]} |t - t_j|.$$

So the interval $[x_k, x_{k+1}]$ contains no point of t_j's except for t_i.

By Theorem 2.3 it follows from (1.3) that

$$\begin{aligned}
&\int_{-1}^{1} |A_{m-2}(\ell_j, x_j; t)|(1+t)u(t)\,dt \\
&\quad = \int_{-1}^{1} A_{m-2}(\ell_j, x_j; t)(1+t)\mathrm{sgn}\,[(t - x_j)\ell_j(u, m; t)]^m u(t)\,dt \\
&\quad = [\mathrm{sgn}\ P_n'(u, m; x_j)^m] \int_{-1}^{1} A_{m-2}(\ell_j, x_j; t)(1+t)[\mathrm{sgn}\ P_n(u, m; t)^m] u(t)\,dt \\
&\quad = [\mathrm{sgn}\ P_n'(u, m; x_j)^m] \lambda_{j,m-2,n}(u, m)(1 + x_j) \\
&\quad = \lambda_{m-2,n}(u, m; x_j)(1 + x_j).
\end{aligned} \tag{3.41}$$

We need an Erdős–Turán inequality ($\ell_0 = \ell_{n+1}$) [3]

$$\ell_k(t) + \ell_{k+1}(t) \geqslant 1, \quad t \in [x_k, x_{k+1}], \ \ k = 0, 1, \ldots, n,$$

from which it follows by (3.21) that

$$\ell_k(t)^m + \ell_{k+1}(t)^m \geqslant 2^{1-m}[\ell_k(t) + \ell_{k+1}(t)]^m \geqslant 2^{1-m}, \quad t \in [x_k, x_{k+1}],$$

$$k = 0, 1, \ldots, n.$$

Thus by (3.30)

$$\int_{-1}^{1} |A_{m-2}(\ell_k, x_k; t)|(1+t)u(t)\,dt + \int_{-1}^{1} |A_{m-2}(\ell_{k+1}, x_{k+1}; t)|(1+t)u(t)\,dt$$
$$\geqslant \frac{1}{(m-2)!}\left(\frac{x_{k+1}-x_k}{5}\right)^{m-2}$$
$$\times \int_{x_k+(x_{k+1}-x_k)/5}^{x_{k+1}-(x_{k+1}-x_k)/5} \left[\ell_k(t)^m + \ell_{k+1}(t)^m\right](1+t)u(t)\,dt$$
$$\geqslant c(x_{k+1}-x_k)^{m-2} \int_{x_k+(x_{k+1}-x_k)/5}^{x_{k+1}-(x_{k+1}-x_k)/5} (1+t)u(t)\,dt$$
$$\geqslant c(x_{k+1}-x_k)^{m-2} \int_{x_k}^{x_{k+1}} (1+t)u(t)\,dt,$$

which, coupled with (3.41), gives

$$(x_{k+1}-x_k)^{m-2} \int_{x_k}^{x_{k+1}} (1+t)u(t)\,dt$$
$$\leqslant c[\lambda_{m-2,n}(u,m;x_k)(1+x_k) + \lambda_{m-2,n}(u,m;x_{k+1})(1+x_{k+1})]. \tag{3.42}$$

If $x_{k+1}-x_k \leqslant \Delta_n(x_k)+\Delta_n(x_{k+1})$, then

$$\left|\left[(1-x_{k+1}^2)^{1/2}\right]^2 - \left[(1-x_k^2)^{1/2}\right]^2\right| = |x_{k+1}^2 - x_k^2| \leqslant 2(x_{k+1}-x_k)$$
$$\leqslant \frac{c}{n}\left\{\left[(1-x_{k+1}^2)^{1/2}+\frac{1}{n}\right]+\left[(1-x_k^2)^{1/2}+\frac{1}{n}\right]\right\}$$

and hence by Lemma 3.12

$$|\Delta_n(x_{k+1}) - \Delta_n(x_k)| \leqslant \frac{1}{n}|(1-x_{k+1}^2)^{1/2} - (1-x_k^2)^{1/2}| \leqslant \frac{c}{n^2}.$$

So $\Delta_n(x_{k+1}) \leqslant c\Delta_n(x_k)$ and

$$x_{k+1}-x_k \leqslant c\Delta_n(x_k).$$

If $x_{k+1}-x_k > \Delta_n(x_k)+\Delta_n(x_{k+1})$, then using (3.19) inequality (3.42) gives

$$\int_{x_k}^{x_{k+1}} (1+t)u(t)\,dt \leqslant \frac{c}{n}\left[W_n(x_k)(1+x_k) + W_n(x_{k+1})(1+x_{k+1})\right]$$

or

$$\int_{x_k}^{x_{k+1}} |t-t_i|^{p_i}(1+t)\,dt \leqslant \frac{c}{n}\left[W_n(x_k)(1+x_k) + W_n(x_{k+1})(1+x_{k+1})\right]. \tag{3.43}$$

We distinguish the cases when $i \in \{1, r\}$ and $2 \leqslant i \leqslant r-1$.

Case 1: $i \in \{1, r\}$. It is enough to treat the case when $i = 1$. In this case using (3.19) inequality (3.43) yields

$$(1+x_{k+1})^{p_1+2} - (1+x_k)^{p_1+2}$$
$$\leqslant \frac{c}{n}\left\{\left[(1+x_{k+1})^{1/2}+\frac{1}{n}\right]^{2p_1+3} + \left[(1+x_k)^{1/2}+\frac{1}{n}\right]^{2p_1+3}\right\}.$$

Again by Lemma 3.12 we have

$$(1+x_{k+1})^{1/2}-(1+x_k)^{1/2}\leqslant\frac{c}{n}.$$

Thus

$$\begin{aligned}x_{k+1}-x_k&=\left[(1+x_{k+1})^{1/2}+(1+x_k)^{1/2}\right]\left[(1+x_{k+1})^{1/2}-(1+x_k)^{1/2}\right]\\&\leqslant\frac{c}{n}\left[(1+x_{k+1})^{1/2}+(1+x_k)^{1/2}\right]\leqslant\frac{c}{n}\left[(1+x_k)^{1/2}+\frac{1}{n}\right]\\&\leqslant\frac{c}{n}\left[(1-x_k^2)^{1/2}+\frac{1}{n}\right]=c\Delta_n(x_k).\end{aligned}$$

Case 2: $2\leqslant i\leqslant r-1$. In this case by (3.19) inequality (3.43) gives

$$\begin{aligned}&\frac{c}{n}\left[\left(|t_i-x_k|+\frac{1}{n}\right)^{p_i}+\left(|t_i-x_{k+1}|+\frac{1}{n}\right)^{p_i}\right]\\&\quad\geqslant\begin{cases}|t_i-x_k|^{p_i+1}+|t_i-x_{k+1}|^{p_i+1}, & t_i\in[x_k,x_{k+1}],\\ \big|\,|t_i-x_k|^{p_i+1}-|t_i-x_{k+1}|^{p_i+1}\big|, & t_i\notin[x_k,x_{k+1}].\end{cases}\end{aligned}\tag{3.44}$$

If $t_i\in[x_k,x_{k+1}]$ then by Lemma 3.12 it follows from (3.44) that

$$x_{k+1}-x_k=|t_i-x_k|+|t_i-x_{k+1}|\leqslant\frac{c}{n}\leqslant c\Delta_n(x_k).$$

If $t_i\notin[x_k,x_{k+1}]$ then by Lemma 3.12 it follows from (3.44) that

$$x_{k+1}-x_k=\big|\,|t_i-x_k|-|t_i-x_{k+1}|\,\big|\leqslant\frac{c}{n}\leqslant c\Delta_n(x_k).$$

Claim 2. $x_{k+1}-x_k\geqslant c\Delta_n(x_k),\ k=0,1,\ldots,n.$

Applying Lemma 3.13 several times and using (3.19), we obtain

$$\begin{aligned}&|A_{m-2}^{(m-1)}(\ell_k,x_k;x)|\delta_{mn}(x)^{m-1}W_{mn}(x)\\&\quad\leqslant cmn\int_{-1}^{1}|A_{m-2}^{(m-1)}(\ell_k,x_k;t)|(1-t^2)^{(m-1)/2}u(t)\,dt\\&\quad\leqslant c(mn)^m\int_{-1}^{1}|A_{m-2}(\ell_k,x_k;t)|u(t)\,dt\\&\quad=c(mn)^m\lambda_{m-2,n}(u,m;x_k)\\&\quad\leqslant cn^{m-1}W_n(x_k)\Delta_n(x_k)^{m-2}.\end{aligned}$$

Thus it follows by Lemma 3.14 from Claim 1 that

$$\begin{aligned}|A_{m-2}^{(m-1)}(\ell_k,x_k;x)|&\leqslant cn^{m-1}W_n(x_k)\Delta_n(x_k)^{m-2}\delta_{mn}(x)^{1-m}W_{mn}(x)^{-1}\\&\leqslant c\Delta_n(x_k)^{-1},\quad x\in[x_k,x_{k+1}].\end{aligned}$$

But by the mean value theorem for the derivatives for some point $\xi\in[x_k,x_{k+1}]$

$$\begin{aligned}1&=A_{m-2}^{(m-2)}(\ell_k,x_k;x_k)-A_{m-2}^{(m-2)}(\ell_k,x_k;x_{k+1})\\&=(x_k-x_{k+1})A_{m-2}^{(m-1)}(\ell_k,x_k;\xi)\leqslant c(x_{k+1}-x_k)\Delta_n(x_k)^{-1}.\end{aligned}$$

Then

$$x_{k+1} - x_k \geqslant c\Delta_n(x_k). \qquad \square$$

Remark 3.2. Theorem 9.20 in [6, pp. 164–165] and Theorem 2 in [11] are special cases of Theorem 3.4 when $m = 2$ and $m \in \mathbf{N}_2$ without the restriction (3.40), respectively. But their proofs (the latter directly cites the former) are not suitable for the case when $p_i < 0,\ 2 \leqslant i \leqslant r-1$. It is still open that if Theorem 3.4 remains true without this restriction.

As a consequence of Theorems 2.6, 3.3, and 3.4 we state the following.

Theorem 3.5. *Let relation* (3.17) *with* (3.40) *prevail. Then for* $j \in \mathbf{M}_1 \setminus \{0\}$

$$\begin{aligned}&|\lambda_{jn}(u, m; x)| \leqslant c\lambda_n(u; x)\Delta_n(x)^j \leqslant \frac{c}{n} W_n(x)\Delta_n(x)^j,\\ &x \in [-1, 1] \setminus [(x_{1n}, x_{2n}) \cup (x_{n-1,n}, x_{nn})].\end{aligned} \tag{3.45}$$

Theorem 3.6. *Let* $m \in \mathbf{N}_2$. *Then relation* (3.17) *is equivalent to*

$$\frac{c_1}{n} W_n(x) \leqslant \lambda_{0n}(u, m; x) \leqslant \frac{c_2}{n} W_n(x). \tag{3.46}$$

Proof. It suffices to show the implication (3.46) $\Longrightarrow$ (3.17). By (3.10) and (3.46)

$$u(x)(1-x^2)^{1/2} \geqslant c \limsup_{n\to\infty} n\lambda_{0n}(u, m; x) = c(1-x^2)^{1/2} \prod_{i=1}^{r} |x - t_i|^{p_i},$$

that is,

$$u(x) \geqslant c \prod_{i=1}^{r} |x - t_i|^{p_i}.$$

Thus $\lim_{n\to\infty} \gamma_n(u)^{1/n} = 2$. Then applying Lemma 3.6 and using (3.46), we obtain

$$u(x) \leqslant c \prod_{i=1}^{r} |x - t_i|^{p_i}. \qquad \square$$

For the Chebyshev weight $v(x)$ Turán raised the following problem [15, p. 47]:

Problem 26. Give an explicit formula for $\lambda_{kjn}(v, m)$ and determine its asymptotic behavior as $n \to \infty$.

The following theorem gives an answer to the same problem for a weight $u \sim W$.

Theorem 3.7. *If* (3.17) *with* (3.40) *is true, then, with the constants associated with the symbol* $\sim$ *depending on u and m,*

$$\begin{cases} [\operatorname{sgn} P_n'(u, m; x_{kn})^m]\lambda_{kjn}(u, m) \sim \frac{1}{n} W_n(x_{kn})\Delta_n(x_{kn})^j, & j \in \mathbf{M}_2,\\ |\lambda_{kjn}(u, m)| \leqslant \frac{c}{n} W_n(x_{kn})\Delta_n(x_{kn})^j, & j \in \mathbf{M}_1 \setminus \{0\}. \end{cases} \tag{3.47}$$

Proof. The first formula in (3.47) follows directly from (3.19) and (2.27); the second one follows from the first one, (3.45), and (2.27). $\square$

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